# Some facts about quotients

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**Thm (Chevalley)**

Homogeneous spaces $G/H$ are always $G$-quasi-projective

**Proof (idea):** ($k = \bar{k}$)

1) Consider $\underline{k}[G]$ as an $H$-rep. Find fin. dim'l $V \subset I_H \subset k[G]$ which generates this ideal

$$\Longrightarrow H = \mathrm{Stab}(I_H) = \mathrm{Stab}(V)$$

not element-wise

2) Find finite dim $V' \supset V$ s.t. $G$ acts faithfully

then still have $H = \mathrm{Stab}(V \subset V')$

3) Plücker embedding Grassmanian $\hookrightarrow \mathbb{P}(\Lambda^{\dim(V)}(V'))$

$$V \longmapsto \Lambda^{top}(V) \subset \Lambda^{\dim(V)}(V')$$

$$H = \mathrm{Stab}(\Lambda^{top}(V)) \text{ in } \mathbb{P}(\Lambda^{\dim(V)}(V'))$$

$\blacksquare$

In fact this scheme structure is the "correct" one

Existence of quotients: We don't quite have the technology yet, but there is a very general

Thm: If $G \times X \hookrightarrow X \times X$ is a monomorphism, then the sheaf $X/G$ is an algebraic space

$\Leftrightarrow G(R)$ acts freely on $X(R)$ for all $R$

define this!

Can take this as motivation for thy of alg. spaces

What is an algebraic space?

Things to discuss: representable morphisms, properties of repr. morphisms, e.g. open immersions

Def: $F : \text{Ring} \to \text{Set}$ such that

0) sheaf

1) $F \xrightarrow{\Delta} F \times F$ is representable by schemes

2) $\exists$ a surjective étale map from a scheme

Equivalence relations in schemes: $R \to X \times X$,

$\boxed{\text{ex:}}$ relation associated to map $X \to Y$

NB, this is necessary

NB, this is not ...

<u>Discuss</u>: construction $U/R$, $\boxed{\text{involving sheafification}}$ (also note fiber products are sheaves)

↳ example of surjective map $X \twoheadrightarrow Y$, then $Y$ is a quotient

Main thm on alg. spaces (Stacks Tag 0455)

TFAE for a sheaf $F : \text{Ring} \to \text{Set}$

1) $F$ alg. space (i.e. $\Delta_F$ repr., $\exists$ etale surj $U \to X$) — scheme

2) $F$ is sheaf, $\exists$ repres. surjection, etale $U \to X$ — scheme

3) $\exists$ equiv. relation $R \to U \times U$ in schemes s.t. $R \to U$ etale, and $F \cong U/R$

easy

Also these are equiv. to the same conditions, but with "flat and finitely presented" instead of etale. [harder]

The point is that $G \times X \to X \times X$ is an equivalence relation and $G \times X \to X$ is fppf (even smooth).

↳ Another way to state the main theorem is that an fppf equivalence relation is "equivalent" to an etale equivalence relation

↳ simply means $(U/R)(A)$ $\forall A \in \text{Ring}$ $\cong$

$W_0 \times_F W_0 \dashrightarrow V_0 \times_F V_0$

$$\begin{tikzcd}
W_0 \times_F W_0 \arrow[rr, dashed] \arrow[dr, shift left] \arrow[dr, shift right] \arrow[dd, dashed] & & V_0 \times_F V_0 \arrow[d, shift left] \arrow[d, shift right] \\
& W_0 \arrow[r] \arrow[d] & V_0 \arrow[d] \\
U_0 \times_F U_0 \arrow[r, shift left] \arrow[r, shift right] & U_0 \arrow[r, two heads, "et"'] & F
\end{tikzcd}$$

$\forall A \in \text{Ring}$ $\cong$ $(U'/R')(A)$

i.e. category of alg. spaces is equivalent to category of equivalence relations in schemes

## Rem on separation axioms:

(Tag 03HJ)

The diagonal $\Delta_{X/Y}$ of a map of alg. spaces $X \to Y$ will be a repres., locally f.t., monomorphism, separated, locally quasi-finite

**Def:** $f: X \to Y$ is separated if $\Delta_{X/Y}$ is closed

quasi-separated if $\Delta_{X/Y}$ is quasi-compact

**Terrible example:** $\mathbb{A}^1_{\mathbb{Q}}/\mathbb{Z}$, $\mathbb{A}^1_{\mathbb{Q}}(R) = R$, $\mathbb{Z}(R) = \mathrm{Map}(\pi_0(\mathrm{Spec}\, R), \mathbb{Z})$

action by addition

this is not quasi-separated. this is a nice class of things to work with

means $\exists\, U \twoheadrightarrow X$ w/ $U$ loc. f.t.

**Thm:** If $X$ is a locally finite type quasi-separated alg. space/$k$, then $\exists$ a dense open subscheme $X' \subset X$.

of $G$ on a l.f.t. scheme

q-proj.

**Cor:** for free actions of $G$ on a l.f.t. scheme, you can at least form a q-proj. quotient for some open subscheme

$$\hookrightarrow \quad \begin{array}{ccc} X'/G & \cong & Y' \\ \| & & \cap \text{ open} \\ X/G & \cong & Y \end{array}$$

**Discuss:**

$G$-torsors,
1) definition $G \circlearrowright X$, $G \times X \xrightarrow{\cong} X \times_Y X$
2) representability: always alg. space
G-torsor over scheme, affine $G$, are schemes

**Cor:** $G$ acts on lft. scheme freely, then $\exists$ dense open subscheme covered by $G$-equiv. open affines

$\hookrightarrow$ quotient space is a scheme iff. it admits a $G$-equiv. open affine cover